ANEXO 4.- FUNCIONES DE BESSEL

La ecuación diferencial: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (a^2 - \frac{n^2}{x^2})\,y = 0$ *, se conoce como ecuación diferencial de Bessel de orden n, cuya solución es:* $\begin{cases} y = C_1\ J_n(x) + C_2\ J_{-n}(x) & para:\ \ n \neq 0, 1, 2, \ldots \\ y = C_1\ J_n(x) + C_2\ Y_n(x) & para:\ \ n = 0, 1, 2, \ldots \end{cases}$ *, en la que* C_1 *y* C_2 *son constantes a determinar;* $J_n(x)$ *y* $Y_n(x)$ *son las funciones de Bessel de primera y segunda especie, respectivamente y orden n, de la forma:*

$$J_n(x) = \frac{x^n}{2^n\ n!} - \frac{x^{n+2}}{2^{n+2}\ (n+1)!} + \frac{x^{n+4}}{2^{n+4}\ 2!\,(n+2)!} - \ldots + (-1)^k\ \frac{x^{n+2k}}{2^{n+2k}\ k!\,(n+k)!} + \ldots$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4\ (2!)^2} - \ldots + (-1)^k\ \frac{x^{2k}}{2^{2k}\ (k!)^2} + \ldots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\ 2!} + \frac{x^5}{2^5\ 2!\,3!} - \ldots + (-1)^k\ \frac{x^{1+2k}}{2^{1+2k}\ k!\,(1+k)!} + \ldots$$

$$Y_n(x) = J_n(x)\log x - x^{-n}\sum_{m=0}^{n-1}\frac{2^{n-2m-1}\,(n-m-1)!\,x^{2m}}{m!} - \frac{x^n}{2^{n+1}n!}\sum_{m=0}^{n-1}\frac{(-1)^n\ k_{nn}\ x^{2n}}{2.4\ldots2n\,(2n+2)(2n+4)\ldots(2n+2n)} =$$

$$= \frac{2}{\pi}\{\ln\frac{x}{2} + \gamma\}J_n(x) - \frac{1}{\pi}\sum_{\mu=0}^{n-1}(n-\mu-1)!\,(\frac{x}{2})^{2\mu-n} - \frac{1}{\pi}\sum_{\mu=0}^{n-1}(-1)^{\mu}\,\Phi(k) + \frac{\Phi(n+k)\,(\frac{x}{2})^{2k+n}}{k!\,(n+k)!}$$

siendo: $\begin{cases} \gamma = 0{,}5772156 \quad (Cte\ de\ Euler) \\ \Phi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \quad ; \quad \Phi(0) = 0 \end{cases}$

Para: $n = 0$: $Y_0(x) = \frac{2}{\pi}\{\ln\frac{x}{2} + g\}J_0(x) + \frac{2}{\pi}\{\frac{x^2}{2} - \frac{x^4}{2^2.4^2}(1 + \frac{1}{2}) + \frac{x^6}{2^2.4^2.6^2}(1 + \frac{1}{2} + \frac{1}{3}) + \ldots\}$

Para: $n = 0, 1, 2, \ldots \qquad Y_{-n}(x) = (-1)^n Y_n(x)$

$$Y_0'(x) = -\,Y_0(x)$$

FUNCIONES DE BESSEL MODIFICADAS

La ecuación diferencial:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - (a^2 - \frac{n^2}{x^2})\, y = 0$$

se conoce como ecuación diferencial de Bessel modificada de orden n, cuya solución es:

$$\begin{cases} y = C_1\ I_n(x) + C_2\ I_{-n}(x) & para:\ \ n \neq 0, 1, 2, \ldots \\ y = C_1\ I_n(x) + C_2\ K_n(x) & para:\ \ n = 0, 1, 2, \ldots \end{cases}$$

$$I_0(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{2^2.4^2} + \ldots$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \ldots$$

$$I'_0(x) = I_1(x)$$

$$K_0(x) = \{ln\ \frac{x}{2} + g\}\, I_0(x) + \{\frac{x^2}{2} + \frac{x^4}{2^2.4^2}(1 + \frac{1}{2}) + \frac{x^6}{2^2.4^2.6^2}(1 + \frac{1}{2} + \frac{1}{3}) + \ldots\}$$

Para: n = 0, 1, 2, ... $\quad K_{-n}(x) = K_n(x)$

FÓRMULAS DE RECURRENCIA DE FUNCIONES DE BESSEL

$$I_{n+1}(x) = \frac{2n}{x}\, I_n(x) - I_{n-1}(x)$$

$$I_{n+1}(x) = -\frac{2n}{x}\, I_n(x) + I_{n-1}(x)$$

$$x\, I'_n(x) + n\, I_n(x) = x\, I_{n-1}(x)$$

$$x\, I'_n(x) - n\, I_n(x) = x\, I_{n+1}(x)$$

$$x\, K'_n(x) + n\, K_n(x) = -x\, K_{n-1}(x)$$

$$x\, K'_n(x) - n\, K_n(x) = -x\, K_{n+1}(x)$$

$$x\, J'_n(x) + n\, J_n(x) = x\, J_{n-1}(x)$$

$$x\, J'_n(x) - n\, J_n(x) = -x\, J_{n+1}(x)$$

$$x\, Y'_n(x) + n\, Y_n(x) = -x\, Y_{n-1}(x)$$

$$x\, Y'_n(x) - n\, Y_n(x) = -x\, Y_{n+1}(x)$$

$$J_n(x)\, Y'_n(x) - Y_n(x)\, J'_n(x) = \frac{2}{\pi x}$$

$$I_n(x)\,K'_n(x) - K_n(x)\,I'_n(x) = -\frac{1}{x}$$

$$I_n(x)\,K_{n+1}(x) + K_n(x)\,I_{n+1}(x) = \frac{1}{x}$$

$$K_{-n}(x) = K_n(x)$$

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2\,x}}\;exp\,(-x)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2\,n\,J_n(x)}{x} \quad ; \quad J_{n-1}(x) - J_{n+1}(x) = 2\,\frac{d}{dx}\,J_n(x)$$

$$\frac{d}{dx}\,\{J_n(x)\} = J_{n-1}(x) - \frac{n\,J_n(x)}{x} \quad ; \quad \frac{d}{dx}\,\{J_n(x)\} = -\,J_{n+1}(x) + \frac{n\,J_n(x)}{x}$$

$$\frac{d}{dx}\,\{J_0(a\,x)\} = -\,a\,J_1(a\,x) \quad ; \quad \frac{d}{dx}\,\{Y_0(a\,x)\} = a\,Y_1(a\,x) + \frac{n\,J_n(x)}{x}$$

$$x\,J_1(a\,x)\Big\rangle_0^R = \int_0^R a\,x\,J_0(a\,x)\;dx \quad ; \quad R\,J_1(a\,R) = a\int_0^R x\,J_0(a\,x)\;dx$$

$$\frac{d}{dx}\,\{x^n\;J_n(x)\} = x^n\;J_{n-1}(x)$$

$$\frac{d}{dx}\,\{x^{n-1}\;J_n(x)\} = -\,x^{-n}\;J_{n+1}(x)$$

$$\frac{d}{dx}\,\{x^n\;J_n(a\,x)\} = a\,x^n\;J_{n-1}(a\,x) \quad \xrightarrow{Para\ n\,=\,1} \quad \frac{d}{dx}\,\{x\,J_1(a\,x)\} = a\,x\,J_0(a\,x)$$

$$\frac{d}{dx}\,\{x^n\;I_n(x)\} = x^n\;I_{n-1}(x)$$

$$\frac{d}{dx}\,\{x^{-n}\;I_n(x)\} = x^{n-1}\;I_{n+1}(x)$$

$$\frac{d}{dx}\,\{I_0(a\,x)\} = a\,I_1(a\,x)$$

$$\frac{d}{dx}\,\{x^n\;K_n(x)\} = -\,x^n\;K_{n-1}(x)$$

$$\frac{d}{dx}\,\{x^{n-1}\;K_n(x)\} = -\,x^{-n}\;K_{n+1}(x)$$

$$\frac{d}{dx}\,\{K_0(a\,x)\} = -\,a\,K_1(a\,x)$$